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[NON-LINEAR OSCILLATIONS OF RAREFIED PLASMA. I.]

by

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from

Nuclear Fusion 1 (2), 82-100, 1961.

Translated by

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Facsimile Price $ [redacted]
Microfilm Price $ [redacted]

Available from the
Office of Technical Services
Department of Commerce
Washington 25, D. C.

U.K.A.E.A. Research Group,
Atomic Energy Research Establishment,
Harwell.
October 1962

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1. Introduction

There is a fairly broad literature on the oscillating properties of a rarefied plasma, including some review articles. ~~those of the survey type.~~ As a rule, however, only linear or small plasma oscillations are studied, that is to say where the type of motion is independent of the amplitude of oscillation /1 - 3/.

In linear theory the oscillating motion of a plasma is represented by the superposition of the individual waves - harmonics without mutual influence ($\exp i(\omega t - \underline{k} \cdot \underline{r})$, where ω is the frequency, $\underline{k}$ is the wave vector).

Generally speaking, there is a certain connection between ω and $\underline{k}$ - "dispersion law" ($\omega = \omega(\underline{k})$).

It is clear that taking into account nonlinearity of the oscillations can change the picture of motion occurring in the linear theory. We will refer to the direct analogy with acoustic oscillations in ordinary gas dynamics. It is well known that acoustic waves, harmonics in linear approximation, are distorted in the process of time on account of the finite amplitude (see Fig. 1). This distortion of the acoustic wave profile is described mathematically by a nonlinear term in the equation of motion $(\underline{v} \cdot \nabla)\underline{v}$ in gas dynamics, reflecting the fact that the sections of the profile corresponding to nigh velocity tend to lead sections with lower velocity. If the sound has a sufficiently high amplitude for this type of nonlinear effect to appear earlier than the wave simply attenuates on account of the thermal conductivity and viscosity in the gas, then ultimately discontinuities are formed.

In ordinary gas dynamics (disregarding dissipation processes) discontinuities are represented by mathematical surfaces representing the fronts of shock waves. In actual fact the front of a shock wave represents a ~~transient~~ transition zone of finite thickness (in ordinary gas dynamics ~~in any case~~ always greater than the free path length). We will now trace the possible distortion of the profile of any given harmonic wave in a rarefied plasma*. Two factors will affect the development of such a wave in the process of time: (1) nonlinearity, (2) absorption. We will first study the influence of nonlinearity, assuming absorption to be small.

The tendency to increasing steepness of the wave front on account of nonlinearity is preserved also in waves in a rarefied plasma (an exception is afforded by any waves with ~~[illegible]~~ transverse polarisation such, for example, as Alfvén magnetohydrodynamic waves). Whilst in gas dynamics the rise in front-steepness is limited by dissipation effects, dispersion effects may play the main part in a rarefied plasma. In our example with the harmonic wave, the "competition" between nonlinearity striving to ~~invert~~ "overturn" the wave, and dispersion may be illustrated in the following manner. The increase in front steepness in the language of Fourier series signifies the ~~origin~~ generation of higher harmonics in the wave ~~and~~ under the influence of nonlinearity. To a first (linear) approximation any wave remains harmonic

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* As distinct from sound in ordinary gas dynamics this presentation of the problem for a plasma is more complicated, since a number of different types of oscillation (oscillation "branches") may exist in a plasma. ~~There~~ For example, seven different branches are possible in a constant magnetic field.

(exp i ($\omega t - \underline{k} \cdot \underline{r}$)); in the second approximation the second harmonics must appear (as occure in the acoustic wave). In the expansion with respect to ~~for~~ wave amplitude the equation for correction of the second approximation takes the form

$$\mathcal{L}_0 f_2 = 1 f_1^2 \exp i(2\omega t - 2\underline{k} \cdot \underline{r}) \qquad (1)$$

Here f is the deviation of some value characterising the plasma or field from the equilibrium value (the ~~indices~~ suffixes 1, 2 signify the first and second approximation respectively); $\mathcal{L}_0$ is a linear operator corresponding to the linear oscillations of the plasma with a dispersion law $\omega = \omega(\underline{k})$. In a certain sense ~~given concept~~, equation (1) has ~~takes~~ the form of the equation of motion of the "oscillator" under the action of the exciting force $\sim f_1^2$. It is clear that the second harmonic will be excited ~~induced~~ if this force is in "resonance" with the "natural frequency" of the oscillator, that is to say if twice the initial frequency 2ω corresponds (in the dispersion law) to the wave vector equal to 2k. Generally speaking, this "resonance" will be achieved only for a linear dispersion law $\omega = ck$, as in the case of ordinary sound. Generally speaking, for any given dispersion law there will be no energy transfer from the main harmonic to the second and so on if the "exciting force" is far removed from "resonance". This qualitative illustration indicates that periodic waves in the plasma, in the frequency range where considerable deviation from the linear dispersion law occurs, can be propagated

A special type of non-distorting wave is represented by what are called "solitary waves", where dispersion also plays an important part. A specific type of dispersion law is necessary to bring about such waves (see section 4).

Without distortion of form due to nonlinearity. Knowing the trend of the dispersion curves $\omega = \omega(\underline{k})$ from the theory of linear oscillations of a plasma, it is thus possible to forecast some general properties of nonlinear motions. Consider, for example, magneto-acoustic oscillations propagated across a magnetic field in a cold plasma. At frequencies below ω_{Hi}, the Larmor frequency of the ions, the phase velocity of such oscillations is equal, to a high degree of accuracy, to $H_0/\sqrt{4\pi\rho_0}$ (H_0 is the unperturbed magnetic field and ρ_0 the mass density of the unperturbed plasma). With increasing frequency, the phase velocity starts to diminish and, on approaching a frequency equal to the mean geometric frequency of the ion and electron frequencies $(\omega_{Hi}\,\omega_{He})^{\frac{1}{2}}$, becomes zero.

The dispersion curve for this type of oscillation is shown in Fig. 2.

$$\frac{\omega^2}{k^2} = \frac{H_0^2\,\omega_0^2/c^2}{4\pi\rho_0(k^2+\omega_0^2/c^2)}\,; \qquad (2)$$

$$\omega_0^2 = 4\pi ne^2/m$$

Deviations from the linear dispersion law start to appear markedly for $k \to \omega_0/c$. The quantity c/ω_0 will be the characteristic spatial scale for stationary nonlinear magneto-acoustic waves (see section 4).

In the absence of a magnetic field, ion acoustic oscillations in a rarefied plasma enter into the linear theory /3/, as is known, if the electron pressure is appreciably in excess of the ion pressure

$p_e \gg p_i$. These conditions occur, for example, in a nonisothermal plasma when the electron temperature is appreciably in excess of the ion temperature. Assuming for simplicity "cold" ions ($T_i = 0$) then the dispersion law will be as in (2)

$$\frac{\omega^2}{k^2} = \frac{T_e}{M} \frac{\varkappa^2}{\varkappa^2 + k^2},$$

$$\varkappa^2 = \Omega_0^2 M/T_e, \quad \Omega_0^2 = 4\pi ne^2/M. \qquad (3)$$

The Debye radius $1/\varkappa$ here takes on the function of the characteristic scale.

Finally, the electron oscillations in the "cold" plasma in the absence of a magnetic field have, as the dispersion relationship, simply $\omega^2 = \omega_0^2$. There is no characteristic scale here and stationary nonlinear waves can be constructed with any desired spatial period/5/. For the nonlinear motions of a cold electron plasma, the strict unidimensional problem is resolved for any initial and boundary conditions if the amplitude is not too great/6/ (see also section 2).

Thus, as distinct from ordinary gas dynamics where dissipation processes are the cause of limiting the further increase in the front steepness of nonlinear motions at some stage, in the rarefied plasma dispersion effects may be the cause.

The difference between these two groups of effects is reflected in the mathematical structure of the basic equations. The dissipation effects (viscosity, thermal conductivity etc.) destroy their reversibility - increase the order of the derivatives by an odd

number (for example viscosity introduces terms with second derivatives into the Euler equation in gas dynamics). Dispersion effects do not destroy reversibility and increase the order of differentiation in equations by an even number.

Up to the present our consideration has been given to waves of finite but small amplitude. For waves with a fairly high amplitude a completely different situation may occur. The dispersion effects may prove inadequate for limiting the increase in front steepness and at amplitudes exceeding some critical value the front may be "inverted". Multivelocity flow regions occur (section 3) (this of course has meaning ~~may doubtless exist~~ in relation to the initially cold plasma).

The specific effects of wave absorption are inherent in the rarefied plasma, as distinct from the gas-dynamic medium (for example "Landau damping")/8/. It is known from linear theory /9-10/ that this type of effect is associated with a given group of particles from the velocity distribution which interacts resonance-wise with the electric and magnetic field of the wave. Such particles are, for example, particles travelling at velocities near to the phase velocity of the wave $v - (\omega/k) = 0$, or particles in whose system of co-ordinates the wave frequency is near to the cyclotron frequency $v - \left[(\omega - \omega_H)/k\right] = 0$. The energy exchange between plasma and wave field depends substantially on the local behaviour of the distribution functions of the particles (ions, electrons) in the vicinity of such "resonance" velocities. Nonlinearities substantially influence the form of the distribution function at these "points" (so-called "background" distortion phenomenon - the background being the equilibrium distribution function).

Footnote to p.6

* The existence of nonlinear undamped waves in a plasma where "background" distortion is represented by what are called "captured" particles, serve as a direct example of this.

The linearised theory does not describe this effect. It is valid therefore only when collisions can restore ~~a balanced~~ the equilibrium particle-velocity distribution. This poses a limitation on the wave amplitude where damping can be described by the linear theory (see sec.5). What is called the "quasi-linear" theory (section 5) represents some advance in the linear theory, taking into account the part played by small nonlinear effects only "once" - it considers "background" distortion. Formally the method for obtaining the quasi-linear equations brings to mind the familiar Van der Pol method from the nonlinear theory of oscillations. The distribution function is represented in the form of the sum of two components: a slowly distorting "background" $f_0(\underline{r},\underline{v},t)$ and a rapidly oscillating component $f_1(\underline{r},\underline{v},t)$, satisfying the linearised kinetic equation. In the equation for f_0 there are retained the mean squares of rapidly oscillating values. Wave absorption can be found by means of quasi-linear approximation equations, taking account of "background distortion".

Even this far-from-trivial concept of the evolution of initial perturbation in a plasma, ~~already qualified~~ in which everything is determined by the rivalry between nonlinear "inversion", dispersion and absorption, can be even further complicated if account is taken of possible instabilities. Section 6 will show simple examples of instability accompanying nonlinear waves when their amplitude exceeds some critical value.

2. Some particular cases of non-stationary nonlinear motions of rarefied plasma

With the adoption of a certain number of restrictions it is possible to obtain ~~strict~~ exact solutions for some types of non-stationary nonlinear motions of rarefied plasma.

The most general approach to a description of its behaviour is based on the use of kinetic equations (taking into account self-consistent electric and magnetic fields) for the velocity distribution functions $f_{e,i}(\underline{r},\underline{v},t)$ of ions and electrons ($\underline{v}$ is the velocity, $\underline{r}$ is the spatial co-ordinate and t the time).

2.1. Electron oscillations of a "cold plasma"

Disregarding ion motion (that is to say considering oscillations at high frequencies) and ignoring the thermal dispersion of electron velocities, then it would seem that the unidimensional problem lends itself to a simple exact solution/6/.

In this case all the electrons at a given point have one and the same velocity, satisfying the equation of motion*

$$m\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}\right) = -eE, \qquad (4)$$

The electron density is determined from the continuity equation

$$\frac{\partial n}{\partial t} + \frac{\partial (nv)}{\partial x} = 0 \qquad (5)$$

and finally, for the field we have the equation

$$\frac{\partial E}{\partial x} = -4\pi(n - N_0)e, \qquad (6)$$

where N_0 is the ion density.

* For simplicity we will limit ourselves to the unidimensional plane case.

The system of equations (4)-(6) can be brought to a simple form, using Lagrange variables a,t, where a is the initial co-ordinate of an electron at point x at the moment of time t. The equation of motion, the continuity equation and the Poisson equation are written in these variables in the following manner:

$$mx_{tt} = -eE \tag{4'}$$

$$n(x_a) = n_0(a) \tag{5'}$$

$$E_a = x_a \cdot 4\pi(N_0 - n)e \tag{6'}$$

(the suffix signifies differentiation with respect to the respective independent variable). Differentiating (4') with respect to a and eliminating E_a and n with the aid of (6') and (5') we obtain the oscillation equation for x:

$$x_{tt} + \omega_0^2 x = \varphi(a) \tag{7}$$

where

$$\omega_0^2 = 4\pi N_0 e^2/m, \quad \varphi(a) = \int da\, 4\pi n_0(a)\, e^2/m,$$

or

$$\xi_{tt} + \omega_0^2 \xi = 0, \quad \xi = x + \varphi(a)/\omega_0^2 . \tag{8}$$

Nonlinear motions of electrons in a cold plasma are thus reduced to harmonic oscillations with the Langmuir frequency ω_0. For example, given the initial conditions for velocity and electron density we can, with the aid of (8), find their motion at all subsequent moments in time at any point in space. At the same time, however, the following circumstances must be borne in mind. In developing (8) it was assumed that the Lagrange co-ordinate x

for each electron is a unique function of its initial co-ordinate a (Fig. 3a). If, for example, the initial electron velocities vary sharply in space (as a function of a), then in the course of some period of time t_1 some electrons may overtake (Fig. 3b) and then outstrip (Fig. 3c) their neighbours, so that in some zone in space $x_1 < x < x_2$ at each point x there will be three groups of electrons travelling at different velocities and having, at the initial moment in time, different co-ordinates a_1, a_2, a_3. Equation (8), like the basic system (4) - (6), describes only the single-component current, occurring in the given case only for $t < t_1$; this is connected with the fact that, in writing the basic equations (4)-(6), it was assumed that, at each point in space, at a given moment in time all the electrons travel at one and the same velocity. For a description of the multicomponent current occurring for $t > t_1$, it is necessary to consider a system of equations of motion and continuity for each group of electrons at the given point x, taking into account the Poisson equation in which the sum of the densities of all the electron fluxes must be written as the electron density.

Investigation of this system is very complicated; simple reckoning, however, can consider that the mutually-penetrating electron fluxes occurring after "intersection" of the trajectories imitate in a way "thermal" velocity dispersion, i.e. there is energy dissipation of ordered oscillations into thermal*.

* An interesting model analysis of "multi-flux" electron motion occurring after "intersection" of the trajectories was carried out with computers by O. BUNEMAN /7/ and J. DAWSON /6/. The results of the calculations show that dissipation actually occurs. The analytical investigation of the nature of the multi-flux motion presents substantial difficulties.

2.2. Motion of a plasma across a magnetic field

Substantial simplifications are also afforded in this case if the Larmor radii of the ions and electrons are sufficiently small. The plasma can then be considered as an aggregate of "quasi particles" - Larmor circles with adiabatically maintained magnetic moments $\mu = \mathcal{E}_\perp / H$ ($\mathcal{E}_\perp$ is the Larmor rotation energy of the charge around the magnetic field line of force). Further, disregarding electron inertia and considering the plasma as quasi-neutral, then the basic system of kinetic equations without collisions and the Maxwell equations can be rigorously reduced (for a motion perpendicular to the magnetic field lines of force) to a form recalling the system of equations of magneto gas dynamics with adiabatic index $\gamma = 2$. For example, for plane unidimensional motion

$$\begin{gathered} \rho\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}\right) = -\frac{\partial p}{\partial x} - \frac{\partial}{\partial x}\frac{H^2}{8\pi} \\ \frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0 \\ \frac{\partial(H/\rho)}{\partial t} + v\frac{\partial(H/\rho)}{\partial x} = 0 \\ p = p_0 (\rho/\rho_0)^2 \end{gathered} \qquad (9)$$

where ρ is the mass density and p the pressure, $p = H\int \mu f dv$).

Employing the system (9), the development of any initial perturbation in the plasma across a magnetic field can be readily traced. For example, perturbations of small amplitude ('magnetic sound') are propagated at a velocity

$$\left(\frac{\partial p^*}{\partial \varrho}\right)^{\frac{1}{2}} = \left(\frac{H_0^2}{4\pi\varrho_0} + \gamma\,\frac{p_0}{\varrho_0}\right)^{\frac{1}{2}} ; \left(p^* = \frac{H^2}{8\pi} + p\right).$$

The nonlinear distortion of the profile shape of a perturbation is most clearly illustrated for one important particular case, when the initial distribution of pressure, magnetic field, velocity and density is such that any three of these quantities can be represented as a function of the fourth. As is known, in ordinary gas dynamics, there is the Riemann solution in the analogous case describing what is termed a ~~plane~~ simple wave of arbitrary amplitude. The velocity as a function of time and the co-ordinate is determined in this solution by the implicit function

$$x = t\left[v \pm c(v)\right] + \chi(v) \qquad (10)$$

where $c^2(v) = (dp^*/dv)/(dp/dv)$, whilst $\chi(v)$ is a function dependent on the initial conditions. According to (10) the current profile will be distorted in such a way that, from a given moment, the solution will become ~~three-figured~~ valued. In actual fact, even before this the basic equations will become ~~unacceptable~~ inapplicable at some stage, since ~~one~~ at least one of the assumptions employed for obtaining the system (9) will be ~~destroyed~~ violated. In the case of a rarefied cold plasma ($p \ll H^2/8\pi$) there are dispersion effects (see (2)). The nature of the stationary* currents is considered in detail in Section 4.

* In ordinary gas dynamics (with small free-path length) the stationary current occurs with ~~interruption~~ discontinuity (shock wave) in a similar situation. On the basis of the described mathematical analogy, the hypothesis emerged regarding the possibility of a collisionless shock wave in a rarefied plasma.

2.3. Motion of "cold" ions in a plasma in the absence of a magnetic field

In a rarefied plasma satisfying the condition $p_i \ll p_e$, as is known from linear theory, longitudinal "ion" acoustic oscillations may be propagated with a dispersion equation (3). Since the phase velocity ω/k of such waves considerably exceeds the mean thermal velocity of ions, it is reasonable to disregard their velocity dispersion. On the other hand with these waves ω/k is substantially less than the thermal velocities of electrons. Hence the electric field of the wave will be quasi-stationary with reference to the electrons. Then, if in the range where the electric potential φ is maximal, the electron velocity distribution is Maxwellian $f \sim \exp(-mv^2/2T)$, then the electron density at a given location will be described by the Boltzmann distribution $n = n_0 \times \exp(e\varphi/T)$.* With these assumptions the unidimensional plane motion will be described by the system of equations

$$M\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}\right) = -e\frac{\partial \varphi}{\partial x}$$

$$\frac{\partial n_i}{\partial t} + \frac{\partial (n_i v)}{\partial x} = 0$$

$$-\frac{\partial^2 \varphi}{\partial x^2} = 4\pi e(n_i - n_0 e^{e\varphi/T}). \qquad (11)$$

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*This corresponds to taking account of what are termed "captured" electrons, since the maximum φ corresponds to the "bottom" of the potential trough (minimum potential energy of the electrons). "Capture" of electrons in the potential trough is generally governed by collisions. In this section it is assumed proposed that the frequency of electron collisions is sufficiently high to validify a Boltzmann distribution.

Here M, v and n_i represent respectively the ion mass, the velocity and density of the ions. For motions with characteristic scale considerably in excess of the Debye radius $(T/4\pi n_0 e^2)^{\frac{1}{2}}$, the plasma can be considered quasi-neutral $n_i = n_0 \exp e\varphi/T = n$ (i.e. in the last equation of the system (11) the term $\partial^2\varphi/\partial x^2$ is omitted). Then eliminating the electric field from the equations we obtain

$$M\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}\right) = -\frac{T\partial n}{n\partial x}$$

$$\frac{\partial n}{\partial t} + \frac{\partial (nv)}{\partial x} = 0. \qquad (12)$$

(12) coincides in form with the equations of isothermal motion ($\gamma = 1$) of ordinary gas dynamics. Here, the front, generally speaking, of any initial perturbation will become steeper and steeper in the course of time, to the point of "inversion". However, before the solution becomes multiple-valued, the validity of the quasi-neutrality assumption is destroyed when, with increasing steepness, the characteristic scale falls to the Debye radius. The charge separation effect leads to dispersion (see equation (3)) and stationary motions can be anticipated with a scale in the order of the Debye radius.

3. Multi-velocity currents in a rarefied plasma

In a rarefied plasma there may be currents for which, in some spatial zone, there are at each point two or more groups of particles travelling at different macroscopic velocities. For a sufficiently low plasma density, the mean time between particle collisions is high

and therefore the forces of friction between the different fluxes are neglibly small. The mutual influence of the particles appertaining to different fluxes is governed in this case exclusively by the self-consistent electric and magnetic field, formed by the charges and currents of the particles of all the groups occurring at the given point in space. In this section we will study some cases of such multicomponent currents.

1. We will first consider the case where the external magnetic field in the plasma is directed along the flux. With a marked inequality between electron and ion temperature $T_e \gg T_i$* and where the conditions of quasi-neutrality are satisfied $n_e = n_i$ the "ion sound" propagation is described by the system of equations of isothermal gas dynamics (12).

Now if by means of (12) we attempt to resolve the problem of motion induced in the plasma by a travelling infinite piston**, then in the case of low piston velocity U it is found that the perturbation occurring takes the form of an ion acoustic wave travelling away from the piston, at a velocity c, whilst the velocity of the particles in the wave is equal to the velocity of the piston, taking into account the retardation: $v = U(x - ct)$. This picture coincides with the corresponding picture in the case of gas dynamics. If, however, we increase the velocity of the piston this analogy

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* Frequently achieved in experimental rigs.

** The part of the piston, in the case of a rarefied plasma, can be taken, for example, by a powerful magnetic field applied at the boundary of the plasma and parallel to this boundary.

between the plasma and gas dynamics ceases. Whereas in the case of gas dynamics the dissipation effects produce a shock wave, in the rarefied plasma, where marked frictional forces are absent, there is a two-velocity current zone near to the piston at high velocity, bordering on the single-velocity current zone. In the case studied by us, two groups of ions travelling at different velocities occur as the result of reflection from the piston; the electrons, being single-velocity, travel in such a way as to compensate the positive ion space charge.

In the two-velocity current range the system (12) must be replaced by the following system of equations describing the motion of two ion fluxes:

$$v_t^{(\alpha)} + v^{(\alpha)} v_x^{(\alpha)} = -c^2 \nabla_x \log \sum_{\alpha=1}^{2} n^{(\alpha)},$$

$$n_t^{(\alpha)} + (n^{(\alpha)} v^{(\alpha)})_x = 0, \quad \alpha = 1,2, \qquad (13)$$

with the boundary condition of mirror reflection at the piston travelling at the velocity U:

$$v^{(1)} - v^{(2)} = 2U, \quad n^{(1)} = n^{(2)} \quad \text{for } x = \int U\, dt. \qquad (14)$$

In the case of uniform motion of the piston at the velocity U the flux densities and velocities in the two-velocity range are determined by the following expressions (satisfying, as can be readily demonstrated, the system (13)-(14)): $n^{(1)} = n^{(2)} = n$; $v^{(1)} = 2U$, $v^{(2)} = 0$, where n is the plasma density in the plasma at rest 1 (Fig. 4). The boundary between the rest zone 1 and the two-velocity current zone 2 moves at twice the velocity of the

piston 2U. The electron velocity in the zone 1 is equal to U and the electron density is 2n; hence the electrostatic-potential discontinuity between zones 1 and 2 is equal to:

$$\Delta\varphi = (T/e)\log 2.$$

Investigation of the stability of the two-velocity current shows that the small fluctuations in zone 2 rise exponentially only when the velocity of relative motion of the two ion fluxes is less than c; otherwise the two-velocity current is stable.

Another example leading to multi-velocity current is the problem of "inversion" of the profile of a high-amplitude ion-acoustic wave. Studying the case of the simple (Riemann) ion-acoustic wave described by equation (10) with $c(v) = (T_e/M)^{\frac{1}{2}} = \text{const.}$, then the moment of wave "inversion" is governed by the conditions

$$\partial x/\partial v = \partial^2 x/\partial v^2 = 0, \text{ i.e. } t = -\chi'(v),\ \chi''(v) = 0.$$

After "inversion" of the ion-acoustic wave in the vicinity of that point of the velocity profile near which inversion occurred, there is formed a three-velocity current zone, i.e. three ion fluxes occur, travelling at different velocities. The zone 2 (see Fig. 5) is bounded by the two single-velocity current zones 1 and 3. In zones 1 and 3 the plasma ion motion is described by the system of equations (12) and in the zone 2 by the system (13) (the index α, numbering the fluxes, in this case takes the values 1, 2, 3).

In the case of very high amplitude ion-acoustic waves $v \gg c$ (Mach number $M = v/c \gg 1$) (10) passes over into the equation

describing free-particle motion:

$$x = vt + \chi(v).$$

This equation describes both the single-velocity current (before inversion) and the motion after inversion when the three-velocity zone appears, varying in the course of time.

In concluding the present section, it should be observed that use of the above assumption regarding quasi-neutrality of the plasma is incorrect when there are high gradients of the quantities characterising the plasma motion. In this case the system of equations (11) must be used, taking into account the deviation from quasi-neutrality, or a similar system for multi-velocity problems.

If for example, at the initial moment of time, we set up in the plasma a perturbation with characteristic dimensions considerably less than the Debye radius $D = \sqrt{T/4\pi ne^2}$, then the electron density n may be considered constant (since Debye screening occurs only at distances much greater than D). In this case it is convenient to describe the ion motion in Lagrange variables a,t. Employing the equation of motion of ions

$$Mx_{tt} = eE,$$

the continuity equation

$$Nx_a = N_0(a)$$

and the Poisson equation

$$E_a/x_a = 4\pi e\left[N - n(a)\right],$$

we obtain for the ion co-ordinate x(a,t) the oscillation equation

$$(x_a)_{tt} + \Omega^2(a)x_a = 4\pi N_0(a)e^2/M \qquad (15)$$

differing from (8) only in the fact that instead of the electron

Langmuir frequency ω_0, in the present case we have the ion Langmuir frequency $\Omega = \sqrt{4\pi n(a)e^2/M}$. It follows from (15) that, in a similar manner to the case of electron Langmuir oscillations, with a sufficiently high initial amplitude there occurs "intersection of the trajectories" of the ions, giving rise to the multi-velocity zone.

2. We will now pass to a study of multi-velocity currents in a rarefied plasma in the presence of a magnetic field; at the same time we will study only plasma motion across the magnetic field.

One of the simplest examples of multi-velocity current in the presence of a magnetic field is the problem* of plasma motion under the action of a piston travelling perpendicular to the magnetic-field lines of force at a constant velocity W, much higher than the Alfvén velocity $V_A = \sqrt{H^2/4\pi n M}$. On account of reflection of the plasma ions from the piston (which may actually be a strong magnetic-field zone), there occurs near to it a zone 1 of two-velocity ion motion, the boundary of which travels at a velocity U. At the same time the electrons, due to their small mass, perform a drift motion, so that their current is one-velocity. At the travelling boundary of the two-velocity and single-velocity current zones, continuity conditions are imposed of the material, energy and momentum fluxes.

The system of equations of two-component magnetogas dynamics in the given case takes the form:

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* Studied by M.A. LEONTOVICH with one of the authors.

$$\partial u^\alpha/\partial t + u^\alpha \partial u^\alpha/\partial x = \Omega_H (v^\alpha - v) - V_A^2 \partial \log H/\partial x,$$

$$\partial v^\alpha/\partial t + u^\alpha \partial v^\alpha/\partial x = -\Omega_H (u^\alpha - u),$$

$$\partial n^\alpha/\partial t + \partial (n^\alpha u^\alpha)/\partial x = 0. \qquad (16)$$

n^α, u^α, v^α are the density and velocity components (along the x,y axes) for ions of flux α ($\alpha = 1, 2$);

$n = \sum_\alpha n^\alpha$ is the total ion density (equal to the electron density);

$u = \sum_\alpha n^\alpha u^\alpha/n$, $v = \sum_\alpha n^\alpha v^\alpha/n$

are the electron velocities along the x,y axes

$\Omega_H = eH/M_i c$, $V_A^2 = H^2/4\pi n M_i$.

Expansion in series for the small parameter equal to the ratio of the Alfvén velocity V_A^0 to the piston velocity W (i.e. to the reciprocal of the Mach number $1/M = V_A^0/W \ll 1$), yielded the solution to the system (16) with the respective boundary conditions at the piston and at the boundary with the rest zone 2.

This solution is valid for the initial stage of motion

$$(\Omega_H t/M)^2 \ll 1.$$

It appeared that at the boundary of the single- and two-velocity zones there is a magnetic field discontinuity (and density discontinuity), equal to

$$H_1 - H_0 = H_0,$$

and the boundary itself travels at a velocity

$$U = 2W$$

(with an accuracy to within terms of higher order in 1/M).

Distribution of the quantities characterising the plasma motion in the two-velocity current zone can be found by means of the following intuitive argument. To a first approximation in 1/M the velocity projections of the ions of the first and second fluxes on the axis are equal to:

$$u_1 = 0, \quad u_2 = 2W . \tag{17}$$

The ion densities in these fluxes are mutually equal and coincide with the ion density in the unperturbed plasma. In order to satisfy the conditions of quasi-neutrality, the electrons (the density of which is equal to n) must travel at a velocity W in the two-velocity current zone 1. The magnetic field, proportional to the electron density, in zone 1 is twice the field in the unperturbed plasma H_0. It may be now observed that electron motion at constant velocity W across the magnetic field with a field strength $2H_0$ is possible only in the presence of a constant electric field $E_y = (1/c)W.2H_0$; this electric field, acting after the boundary has passed a given point x on ions previously at rest, gives them a velocity in the direction of the axis y, after the period of time $t - (x/2W)$, equal to

$$v^1 = \left(t - \frac{x}{2W}\right)\frac{e}{M_i} E_y = \Omega_H(2Wt - x). \tag{18a}$$

In studying the ions of the second flux it is necessary to consider that in the course of time $t_2 = (x/W) - t$ they have travelled along the x-axis with a velocity 2W and prior to this during the time $t_1 = t - (x/2W)$ their velocity u^1 was equal to zero and throughout the time $t_1 + t_2$ they were acted upon by the field E_y. Hence the velocity projection of ions of the second flux on the y-axis is equal to

$$v^2 = t_1 \frac{e}{M_i} E_y + t_2 \left(\frac{e}{M_i} E_y - \frac{2W}{c} \frac{e}{M_i} 2H \right) \qquad (18b)$$

$$= \Omega_H (4wt - 3x).$$

The expressions for velocities governed by equations (17)-(18) represent the first expansion terms of the solution to system (16) according to the inverse powers of the Mach number.

It is thus seen that the piston problem, which in the conditions of ordinary and magneto gas dynamics leads to the appearance of shock waves, has a qualitatively different solution in a rarefied plasma; at high piston velocities there is a two-velocity current zone bounded by the piston and a single-velocity zone.

Investigation of high-amplitude waves travelling in a rarefied plasma across a magnetic field also points to the possibility of "inversion" of the wave profile and the occurrence of a three-velocity zone. For motions with characteristic spatial scale L, considerably in excess of $\lambda_0 = c/\omega_0$ (where ω_0 is the electron Langmuir frequency) the equations describing the propagation of such a transverse "magneto-acoustic" wave coincide, as shown in section 2, with the

gas dynamic equations for $\gamma = 2$; if L is in the order of λ_0, the problem is more complicated. However, with a high wave amplitude there is in this case also "intersection of the trajectories" and the formation of a multi-velocity current.

The inversion process is described particularly simply in Lagrange co-ordinates for the case $L \ll \lambda_0$. Here the equation of motion of the particles

$$x_{tt} = \frac{e}{M} E(a)$$

shows that there is uniformly accelerated motion with an acceleration governed by the initial distribution of the magnetic field.

Concluding this section it may be observed that, although a number of particular, similarity or stationary solutions can be found for the system of equations for multi-velocity current of the type (13) or (16)*, the investigation of the general multi-velocity current problem encounters considerable difficulties.

4. Some types of stationary nonlinear motion

Stationary nonlinear waves of various types in a plasma without damping have been investigated by several authors. The standard procedure for obtaining such solutions consists of selecting, in the basic equations, the system of co-ordinates travelling together with the wave. In such a system of reference the time-dependence disappears and the problem is reduced to investigating the stationary

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* The system of equations (13) has, for example, a class of solution of the simple (Riemann) wave type, in which the quantities are interrelated by functional relations.

current. The wave velocity u at first enters the problem as a free parameter. Then, from the condition of resolvability, the boundaries are determined within which u can vary and also the relationship of u with the wave amplitude. We will consider some particular types of stationary wave.

4.1. Stationary ion waves with $p_i \ll p_e$.

Here we shall take stationary motions described by the system of equations (11). Assuming that all the quantities are functions of x and t only in the combination $\xi = x - ut$, the system (11) can be reduced to the one second-order differential equation for the potential φ:

$$\varphi'' = 4\pi n_0 e\left[u/\sqrt{u^2 - 2e\varphi/M} - \exp(e\varphi/T)\right]. \qquad (19)^{**}$$

Integrating (19) once we obtain

$$-\tfrac{1}{2}(\varphi')^2 = 4\pi n_0 e\left(-\frac{uM}{e}\sqrt{u^2 - \frac{2e\varphi}{M}} - \frac{T}{e}\exp\frac{e\varphi}{T}\right) + C. \qquad (20)$$

Depending on selection of the integration constant C, we can construct different periodic waves (see integral curves in the phase plane, Fig. 7).

A special case is represented by C selected from the condition $\varphi' \to 0$ with $\varphi \to 0$, i.e. $C = 4\pi n_0(Mu^2 + T)^*$. This case is isolated separately on the phase plane. It corresponds to what is termed a solitary wave (Fig. 8), representing a symmetrical potential "hump".

** The prime signifies differentiation with respect to ξ.

* Such a choice of C makes sense if we are interested in the asymptotic form of the travelling perturbation, initially localised in a confined spatial zone.

The rate of propagation of this type of wave u as a function of φ_{max} - potential amplitude - is found from (20), assuming $\varphi = 0$ with $\varphi = \varphi_{max}$

$$u^2 = \frac{T}{2M} \frac{\left[\exp(e\varphi_{max}/T) - 1\right]^2}{\exp(e\varphi_{max}/T) - 1 - (e\varphi_{max}/T)} . \qquad (21)$$

In the limiting case of small amplitudes ($e\varphi_{max} \ll T$) u tends towards the velocity of isothermal sound ($\sqrt{T/M}$). At high amplitudes ($e\varphi_{max} \gg T$) $u \approx \frac{1}{2}\sqrt{T/M}\exp(e\varphi_{max}/T)$. For small amplitudes the potential profile in the solitary wave takes the form

$$\varphi \approx \frac{3}{2}\frac{T}{e}\left(1 - \frac{T}{Mu^2}\right) \times \operatorname{sech}^2 \left\{ \frac{\sqrt{\pi n_0 e}}{\sqrt{T}} \sqrt{1 - \frac{T}{Mu^2}}(x - ut) \right\}. \qquad (22)$$

4.2. Stationary magneto-acoustic waves of finite amplitude, propagated across a magnetic field [12-15]

In this case dispersion effects are associated with deviation from the quasi-hydrodynamic equations (9), which disregarded, in the first place, electron inertia and secondly deviation from quasi-neutrality. On taking account of either one of these factors stationary motions may be obtained, differing from the plane-parallel flux. There are two characteristic lengths corresponding to these. The first is evidently associated with the electron inertia (depending on the electron mass). If $c/\omega_0 \gg H_0/(4\pi ne)$* then

* This condition can be re-formulated in the following manner: $nmc^2 \gg H^2/8\pi$.

with increasing front steepness the first factor starts to enter earlier and charge separation can be disregarded. Considering this case we will not yet take into account thermal motion, so that the system of equations describing the motion of ions and electrons and the field profile in the stationary wave takes the form:

$$Mv'_{xi}\,(v_{xi} - u) = eE_x + (e/c)\,v_{yi}H$$

- the equation of motion of the ions in the direction of the x-axis (in which direction the wave is travelling);

$$Mv'_{yi}(v_{xi} - u) = eE_y - (e/c)v_{xi}H$$

- along the y-axis, perpendicular to the direction of travel of the wave and the magnetic field (see Fig. 9);

$$mv'_{xe}(v_{xe} - u) = eE_x - (e/c)v_{ye}H;$$

$$mv'_{ye}(v_{xe} - u) = -eE_y + (e/c)v_{xe}H; \qquad (23)$$

$$n_i = \frac{n_0 u}{u - v_{xi}}; \qquad n_e = \frac{n_0 u}{u - v_{xe}};$$

$$n_i = n_e$$

(From the latter equation, expressing quasi-neutrality, there follows equality of the electron and ion velocity components along x)

$$E_y = (u/c)(H - H_0)$$

$$-H' = (4\pi/c)\,ne(v_{yi} - v_{ye})$$

Eliminating from these equations all the variables other than H, we obtain (with an accuracy to within terms $\sim$ m/M)

$$- \frac{mc^2}{4\pi n_0 e^2 u} \frac{d}{d\xi}\left[\frac{dH}{d\xi}\left(\frac{H^2 - H_0^2}{8\pi n_0 M u} - u\right)\right] - \left(\frac{H^2 - H_0^2}{8\pi n_0 M u} - u\right)$$

$$= \left(\frac{H^2 - H_0^2}{8\pi n_0 M u} - u\right) H + u H_0 \qquad (24)$$

This equation governs the profile of the H variation in the investigated stationary wave. Integrating once, it is brought to the form

$$- a^2 (H')^2 \left(\frac{H^2 - H_0^2}{8\pi n_0 M u} - u\right)^2 = \frac{(H^2 - H_0^2)^2 - 16\pi n_0 M u^2 (H - H_0)^2}{16\pi n_0 M} + C.$$

$$\left(a^2 = \frac{mc^2}{4\pi n_0 e^2} = \frac{c^2}{\omega_0^2}\right) \qquad (25)$$

This equation, like equation (20), describes periodic waves of finite amplitude. The behaviour of the integral curves on the phase plane (H', H) is very reminiscent of the picture already illustrated in Fig. 7. Selecting C from the condition $H' \to 0$ with $H \to H_0$, we obtain the particular case of the solitary wave. From the condition $H' = 0$ with $H = H_{max}$, we find the connection between the wave amplitude and velocity

$$u^2 = \frac{(H_{max} + H_0)^2}{16\pi n_0 M} \qquad (26)$$

For small amplitudes ($H_{max} \to H_0$) the wave velocity, diminishing, ~~strives~~ tends towards $H_0/\sqrt{4\pi n_0 M}$. The magnetic field profile in the case of small amplitude takes the form

$$H = H_0 \left[1 + 2 \left(\frac{4\pi n_0 M u^2}{H_0^2} - 1 \right) \right.$$
$$\left. \times \operatorname{sech}^2 \left[\frac{x}{a} \sqrt{\frac{4\pi n_0 M u^2}{H_0^2} - 1} \right] \right] . \qquad (27)$$

Equation (25) has a real solution ~~for~~ only for ~~not such favourably~~ at arbitrary high values of u and H. For example, for solitary waves solutions exist for $H_{max} < 3H_0$ (i.e. $u < 2H_0/\sqrt{4\pi n_0 M}$). As the wave amplitude approaches the critical, the ion (electron) density ~~strives towards~~ tends to infinity at the wave crest. Physically, this signifies the following. The solitary wave represents an electric potential ~~"bulge"~~ "hump" φ. In the system of co-ordinates associated with the wave, the ion flux from $x = \infty$ travels towards this potential barrier with a velocity u. For amplitudes which are not too high, the initial kinetic energy of the ion $Mu^2/2$ exceeds the height of the potential barrier $e\varphi_{max}$ and the ions, somewhat retarded, pass ~~through~~ over it. However, as ~~emerging~~ follows from the solution, with increasing wave amplitude the potential barrier becomes so high that $e\varphi_{max} > Mu^2/2$. The moment $e\varphi_{max} = Mu^2/2$ corresponds to the amplitude $H_{max} = 3H_0$, $u = 2H_0/\sqrt{4\pi n_0 M}$ (in other words the critical "Mach number" is equal to 2). At the peak of such a wave the ions, having lost velocity, are "checked", and their density rises to infinity. With still greater

amplitudes the ions would be simply "reflected" from the barrier, but the motion corresponding to this concept is not described within the framework of our basic system of equations (23), since after reflection the current becomes "multiflux" (interpenetrating fluxes of incident and reflected ions).

Hence it is seen that dispersion effects cannot stop the "inversion" of waves with a sufficiently high amplitude in a cold plasma.

If we were to consider thermal dispersion of ion velocities, then even with small wave amplitudes we could, generally speaking, find ions being reflected from the barrier (these being ions with small relative velocity $u - v_x$), that is to say, travelling initially in the direction of wave propagation at velocities in the vicinity of u; these ions can be called "captured".

Disregarding "captured" ions, a solution can readily be found for the solitary wave also in the more general form taking into account thermal dispersion, considering the ion velocity distribution function. We will immediately consider the opposite limiting case $M_0 = c^2 \ll H^2/8\pi$, where dispersion is governed by charge separation and not by electron inertia. The ion distribution function $f(v, \xi)$ satisfies the equation

$$(v_x - u)\frac{\partial f}{\partial \xi} - \frac{e\varphi'}{M}\frac{\partial f}{\partial v_x} = 0. \qquad (28)$$

We have not taken into account the effect of the magnetic field on the ions. This is justifiable if the characteristic dimension of the perturbed zone (wavelength) is substantially less than the radius of curvature of the ions in the magnetic field, i.e.

$\frac{H}{4\pi n e} \ll \frac{u}{eH/Mc}$; which is always satisfied, since we assumed $H^2/8\pi \ll nMc^2$ ($u \sim H/\sqrt{4\pi n M}$). For the same reason, in (28) the term is omitted taking account of the action of the rotational electric field E_y, since $E_y = (u/c)(H - H_0)$.

The electric field satisfies the Poisson equation

$$-\varphi'' = 4\pi e(n_i - n_e), \tag{29}$$

where $n_i = \int f \, d\, v$. Since electron inertia is disregarded and consequently the magnetic field ~~migrates~~ is transported at the velocity of the electron motion, the electron density varies in direct proportion to the value of the magnetic field strength:

$$\frac{n_e}{H} = \frac{n_0}{H} \tag{30}$$

Finally, the magnetic field is determined from the Maxwell equation curl $\underline{H} = (4\pi/c)\,\underline{j}$ (the displacement current is immaterial, since it was assumed that $H_0/\sqrt{4\pi n_0 M} \ll c$). This equation yields

$$-H' = -\frac{4\pi}{c} e n v_{ye}, \tag{31}$$

where $v_{ye}c = (\varphi'/H)$ (since, disregarding inertia, $\underline{v} = c\underline{E} \times \underline{H}/H^2$).

The system of equations (28)-(31) is integrated in the following manner. Employing equation (31), the solution is sought to (28) for the distribution function

$$f = f_0\left[u - \sqrt{(v_x - u)^2 + \frac{(H - H_0)H_0}{2\pi n_0 M}}\right],$$

where $f_0(v_x)$ is the ion velocity distribution in the region unperturbed by the wave *. Eliminating the remaining variables apart from H, we obtain

$$-H'' = (4\pi e)^2 \frac{n_0}{H_0} \int\int f_0 \; x$$

$$\left[U - \sqrt{(v_x - u)^2 + \frac{(H - H_0)H_0}{2\pi n_0 M}}\right] dv_x - \frac{n_0}{H_0} H \Bigg\}. \quad (32)$$

Hence the problem is reduced to the solution of an ordinary second-order differential equation with boundary condition $H' \to 0$ as $H \to H_0$. If for example we again revert to the case of cold ions, when $f_0(v_x) = n_0 \delta(v_x)$, then (32) takes the form

$$-H'' = (4\pi e)^2 \frac{n_0}{H_0} \left(\frac{u}{\sqrt{u^2 - \frac{(H - H_0)H_0}{2\pi n_0 M}}} - \frac{H}{H_0} \right). \quad (33)$$

After simple integration we obtain

$$\frac{1}{2} H'^2 = \frac{(4\pi e)^2 n_0^2}{H_0^2} \cdot \frac{H^2}{2}$$

$$+ 2\pi \frac{(4\pi e)^2}{H_0^2} n_0^3 uM \sqrt{u^2 - \frac{(H - H_0)H_0}{2\pi n_0 M}} + C. \quad (34)$$

The solitary wave corresponds to selection of $H' \to 0$ with $H \to H_0$. The characteristic length is $H_0/(4\pi n_0 e)$. The connection

* The absence of "captured" particles means that $f_0(v) = 0$ with $v \geq u - \sqrt{\frac{(H_{max} - H_0)H_0}{2\pi n_0 M}}$

between wave velocity u and magnetic field amplitude H_{max} has the already familiar form $u^2 = (H_{max} + H_0)^2/(16\pi n_0 M)$ as in the opposite limit case (see equation (26)). For an arbitrary distribution function $f_0(v)$ (within the framework of the imposed restriction regarding the absence of captured particles) the connection between u and the wave amplitude is expressed in the following equation:

$$8\pi M \int_{\sqrt{\beta}}^{\infty} v\sqrt{v^2 - \beta}\, f_0(u - v)dv + H^2_{max}$$

$$= 8\pi M \int_{0}^{\infty} v^2 f_0(u - v)dv + H_0^2, \qquad (35)$$

where $$\beta = \frac{(H_{max} - H_0)H_0}{2\pi n_0 M}.$$

Assuming $\langle v^2 \rangle = \int v^2 f\, d\, v$ to be small in comparison with $H_0^2/(4\pi n_0 M)$, then expanding (31) into a series we find

$$u \approx u_0 + \frac{\langle v^2 \rangle \left[(u_0^2 - \beta)^{\frac{3}{2}} - u_0^3 + \frac{3}{2} u_0 \beta\right]}{(2u_0^2 - \beta - 2u_0\sqrt{(u_0^2 - \beta)}(u_0^2 - \beta)} + \ldots, \qquad (36)$$

where $$u_0 = \frac{H_{max} + H_0}{2\sqrt{4\pi n_0 M}}.$$

It is to be remembered that any given solitary waves represent a special type of stationary nonlinear wave. Whereas periodic waves can exist with, generally speaking, an arbitrary dispersion law (only not linear), the existence of solitary waves demands a definite

type of dispersion law. This is connected with the fact that, whilst the spectral decomposition of periodic waves contains discrete ω and k, decomposition of the solitary wave profile has a continuous spectrum. Hence the arguments employed in the introductory section are now unacceptable. We will depict the velocity profile (along the direction of propagation) in the solitary wave (Fig. 10a). It is apparent that sections with the maximum spatial gradients ("effective" k) correspond to the maximum velocity amplitudes. It is clear that the existence of a stationary picture requires that larger k values should correspond to smaller phase velocities ω/k in the linear theory. Then generally speaking the greater influence of the nonlinear term $(v.\nabla)v$ on the sections with the higher velocity amplitude will be compensated by the reduced ω/k*. The magneto-acoustic waves studied by us and the "ion" sound in a plasma with $p_i \ll p_e$ possess this type of dispersion law (ω/k falls with increasing k).

In ordinary hydro dynamics, waves on the surface of a heavy liquid in a channel of finite depth are characterised by a like dispersion law. The theory of waves on "shallow water" displays a fundamental similarity with the theory of isentropic plane motions in ordinary gas dynamics (effective adiabatic index $\gamma = 2$), and, consequently, also with the dynamics of a rarefied plasma travelling across a magnetic field (system of equations (9)).

* Similar considerations would show that nonlinear solitary "rarefaction" waves can be compared with a linear dispersion law of the inverse type (ω/k is constant with small values of k and increases with large values of k)(Fig. 10b).

On "shallow water" the, generally speaking, arbitrary profile of the initial perturbation tends to "invert". However as the front steepness increases, starting from some moment in time, the finiteness of the channel depth becomes important. Taking account of this yields the necessary dispersion effect. The dispersion law for small oscillations on water of finite depth h takes the form

$$\omega^2 = gk \tan h(kh) \qquad (37)$$

where g is the gravitational acceleration. For small values of k (or large wavelengths $\lambda \gg h$), (37) takes the form

$$\left(\frac{\omega}{k}\right)^2 \approx gh \frac{1}{1 + \frac{1}{3}(kh)^2} . \qquad (38)$$

The dispersion relation (38) resembles, for example, the dispersion law (3) for magneto-acoustic waves in a rarefied plasma. The part of $1/\varkappa$ in (3) is taken by the characteristic scale h.

5. "Quasi-linear" approximation

The influence of slight nonlinearity on the absorption effect can be conveniently described within the framework of "quasi-linear" approximation, taking account of the inverse effect of plasma oscillations on the mean particle velocity distribution ("mean background"), but disregarding the nonlinear effects of wave form distortion (the interaction of harmonics). It is clear that this is valid only for sufficiently weak waves.

The second restriction involves the number of simultaneously occurring different harmonics in the wave (in other words the width of the wave packet), in order that resonance particle capture can

be ignored. In the case, for example, of longitudinal Langmuir oscillations it is necessary for this purpose to satisfy the inequality

$$\Delta v_\varphi \equiv \Delta(\omega/k) \ll \Delta \sqrt{e\varphi_0/m}$$

where Δv_φ is the phase velocity dispersion in the wave packet, and φ_0 is the electrostatic potential amplitude in the wave. A similar evaluation can be written for any type of wave in the plasma. Disregarding captured particles permits the mean background to be considered as uniform in space.

5.1. Development of the quasi-linear approximation equation

In this section we will obtain the quasi-linear kinetic equation for the particle velocity distribution function, averaged over the period of oscillations. For simplification we will confine ourselves to two cases - Langmuir electron oscillations in a plasma in the absence of magnetic field and the electromagnetic wave with circular polarisation travelling along a homogeneous magnetic field in a plasma at frequencies near the cyclotron frequency of electrons.

(a) Langmuir oscillations

We will represent the electric field of the oscillations in the form:

$$\underline{E}(\underline{x},t) = \frac{1}{V}\sum \underline{E}_{\underline{k}} \exp\left\{i(\underline{k}\cdot\underline{x} - \omega t)\right\}$$

$$\underline{E}_{-\underline{k}} = \underline{E}^{+}_{\underline{k}}, \quad \mathrm{Re}\,\omega_{-\underline{k}} = -\mathrm{Re}\,\omega_{\underline{k}} \cdot \qquad (39)$$

Expanding the particle distribution function into a Fourier integral for the spatial co-ordinates and for time

$$f = f_0 + \frac{1}{V}\sum_{\underline{k}} f_{\underline{k}} \exp\left\{i\,(\underline{k}\cdot\underline{x} - \omega t)\right\} \qquad (40$$

and assuming a small $f_{\underline{k}}$, we obtain from the linearised Boltzmann equation, disregarding collisions:

$$f_{\underline{k}} = \frac{(e/m)\underline{E}_{\underline{k}} \cdot (\partial f_0/\partial \underline{v})}{-i\omega + i\underline{k}\cdot\underline{v}}, \qquad (42)$$

where f_0, as distinct from the ordinary linear theory, is assumed to be not a constant function but slowly varying in time.

On the other hand, the equation obtained from the kinetic equation by averaging is valid for f_0:

$$\frac{df_0}{dt} = \frac{e}{m}\int \frac{d\underline{k}}{(2\pi)^3}\,\underline{E}^{*}_{\underline{k}} \cdot \frac{\partial f_{\underline{k}}}{\partial \underline{v}}$$

$$= \frac{e^2}{m^2}\,\frac{\partial}{\partial v_i}\int \frac{d\underline{k}}{(2\pi)^3}\left|\underline{E}_{\underline{k}}\right|^2 \frac{k_i k_e}{k^2} \cdot \frac{\partial f_0/\partial v_e}{-i\omega + i\underline{k}\cdot\underline{v}}. \qquad (43)$$

It is known that the integral can be taken for ~~real~~ real k if $\omega = \omega_r + i\omega_I$ and $i\omega_I > 0$. Therefore

$$\frac{df_0}{dt} = \frac{e^2}{m^2}\,\frac{\partial}{\partial v_i}\int \frac{d\underline{k}}{(2\pi)^3}\left|\underline{E}_{\underline{k}}\right|^2 \frac{k_i k_e}{k^2}$$

$$\times \frac{\partial f_0/\partial v_e}{(\omega_r - \underline{k}\cdot\underline{v})^2 + \omega_I^2}\left[i\,(\omega_r - \underline{k}\cdot\underline{v}) + \omega_I\right]$$

$$\cong \pi\,\frac{e^2}{m^2}\,\frac{\partial}{\partial v_i}\int \frac{d\underline{k}}{(2\pi)^3}\left|\underline{E}_{\underline{k}}\right|^2 \frac{k_i k_e}{k^2}\,\frac{\partial f_0}{\partial v_e}\,\delta(\omega_{\underline{k}} - \underline{k}\cdot\underline{v}), \qquad (44)$$

where it was assumed that $\omega_I \ll \omega_r$. In actual fact the presence of an imaginary component in the frequency yields a finite width of interaction zone between the individual Fourier wave harmonic and the plasma particles (a type of "radiation" width).

The energy exchange between plasma and wave is governed by the expression:

$$\dot{\mathcal{E}} = \int \frac{d\underline{k}}{(2\pi)^3} \underline{j}_{\underline{k}} \underline{E}_{\underline{k}}^{+} = - e \int \frac{d\underline{k}}{(2\pi)^3} \int \underline{v} \, f_{\underline{k}} \underline{E}_{\underline{k}}^{+} \, d\underline{v} \qquad (45)$$

or for the individual Fourier component:

$$\dot{\mathcal{E}}_{\underline{k}} = - e \int \underline{v} \, f_{\underline{k}} \underline{E}_{\underline{k}}^{+} \, d\underline{v} . \qquad (46)$$

Writing $f_{\underline{k}}$ from (42) in (46) we obtain:

$$\dot{\mathcal{E}}_{\underline{k}} = - \frac{e^2}{m^2} \underline{E}_{\underline{k}} \int \underline{v} \, \frac{\underline{E}_{\underline{k}}^{+} \partial f_0 / \partial \underline{v}}{- i\omega + i\underline{k} \cdot \underline{v}} \, d\underline{v} .$$

Introducing a cylindrical system of co-ordinates with axes $v_\parallel$ along the k vector, we obtain:

$$\dot{\mathcal{E}}_{\underline{k}} = \pi \frac{\omega_0^2}{k^2} \omega_k F' \Big]_{kv = \omega_k} \cdot \frac{E_k^2}{4\pi N} , \qquad (47)$$

where $F = 2\pi \int f(v_\parallel, v_\perp) \, v_\perp \, dv_\perp$ is the "longitudinal" particle distribution function.

Up to the present we have studied a rarefied plasma disregarding particle collisions. Generally speaking, the influence of collisions is comparable with the influence of a wave on particle motion only when the wave is ~~balanced~~ an equilibrium one /11/, that is to say when its amplitude does not exceed the amplitude of the corresponding harmonic in

the thermal noise spectrum of the plasma. Only for thermal noise are such processes comparable as Čerenkov emission of waves by travelling particles, emission accompanying collisions, "Landau damping" and absorption accompanying collisions. The thermal noise level of plasma oscillations is determined precisely by their mutual balance.

However, for large amplitude oscillations also ("epithermal"), collisions may govern a number of processes and principally resonance absorption. Under the influence of the wave there is severe distortion of the distribution function in the resonant particle zone. As the result of collisions the Maxwell distribution function is partly "restored" and stationary wave absorption is established. At the same time, other effects of collisions ("collision" width in (44), mutual collisions of resonant particles etc.) are small as before. Formally, the equation describing the behaviour of the mean distribution function in time is obtained as a zero term in the expansion of the exact kinetic equation in powers of $1/N_D$ (ratio Ψ/ω_0 or thermal noise energy to thermal energy of the plasma):

$$\frac{df_0}{dt} = \pi \frac{e^2}{m^2} \int d\,k \left|E_k\right|^2 \frac{k_i k_e}{(2\pi)^3 k^2} \frac{\partial f_0}{\partial v_e} \delta(\omega_{\underline{k}} - \underline{k} \cdot \underline{v}) - \sum \underline{St}\,(f_0, f_\alpha). \qquad (48)$$

where the last term describes resonant particle collisions with other plasma particles. This term can, for example, be taken in the Landau form and linearised, since there are few resonant particles:

$$2\mathcal{L}\frac{\partial}{\partial v_i}\frac{1}{v^3}\left\{v_i f_0 + \left[v^2\delta_{ik} - v_i v_k - \frac{T}{m}\frac{v^2\delta_{ik} - 3v_i v_k}{2v^2}\right]\frac{\partial f_0}{\partial v_k}\right\},$$

where $\mathcal{L} = \lambda\omega_0^4/N$ and $\lambda/8\pi$ represents the Coulomb logarithm.

(b) Transverse oscillations in the presence of a constant magnetic field

Different types of wave may be propagated in a plasma situated in a magnetic field. For simplicity we will limit ourselves to a study of the interaction of a plasma with a transverse wave with circular polarisation in the direction of rotation of the electrons and travelling along a homogeneous external field. Introducing cylindrical co-ordinates with the z axis along the external field H, we obtain the following expression for the correction to the distribution function:

$$f_k = iE_- \frac{1}{\omega}\frac{(\omega - kv_\parallel)\frac{\partial f_0}{\partial v_\perp} + v_\perp k\frac{\partial f_0}{\partial v_\parallel}}{\omega - k v_\parallel - \omega_H}\frac{e}{2m}e^{i\varphi}, \qquad (49)$$

where $E_- = E_x - iE_y$, $\omega_H = eH/mc$, and φ is the angle in velocity space. Here we use the fact considered that

$$H_\pm = \pm\frac{ic}{\omega}kE_\pm. \qquad (50)$$

The dispersion equation for this type of wave takes the form:

$$\omega_k = \omega_H(1 - \omega_0^2/k^2c^2). \qquad (51)$$

Writing (49) in the time-averaged Boltzmann equation, taking account of the assumptions adopted, we obtain:

$$\frac{df_0}{dt} = \mathcal{L}_1 \left(\frac{1}{v_\perp} \frac{\partial}{\partial v_\perp} \frac{v}{\perp} + \frac{v_\perp k}{\omega_H} \frac{\partial}{\partial v_\parallel} \right) \times \left| \frac{\Theta(k)}{\left| \omega' - \frac{\omega - \omega_H}{k} \right|} \right|_{k = v_\parallel/(\omega - \omega_H)} \times \left(\frac{\partial}{\partial v_\perp} + \frac{v_\perp k}{\omega_H} \frac{\partial}{\partial v_\parallel} \right) f_0 - \sum_\alpha \underline{St}\,(f_0, f_\alpha), \qquad (52)$$

$$\mathcal{L}_1 = \frac{e^2 E_0^2}{8m^2\eta}; \quad |E_k|^2 = E_0^2 \frac{\Theta}{2\eta}; \quad \frac{1}{2\eta} \int_{-\infty}^{+\infty} \Theta(k)\,dk = 1; \quad \omega' \equiv \frac{d\omega}{dk}.$$

The energy variation of the k-th harmonic is calculated as for Langmuir oscillations. Consequently we obtain:

$$\dot{\mathcal{E}}_k = \frac{\pi^2 e^2}{2mk} |E_k|^2 \frac{\omega_H}{\omega} \int_0^\infty \left(\frac{\partial f_0}{\partial v_\perp} + \frac{v_\perp k}{\omega} \frac{\partial f_0}{\partial v_\parallel} \right) v_\perp^2 dv_\perp. \qquad (54)$$

5.2. Wave absorption in a plasma

(a) In the absence of a magnetic field

Langmuir waves in a rarefied plasma are damped if the resonant particle distribution function decreases, i.e. $\partial F/\partial v < 0$ for $v = \omega_k/k$. This is associated with the fact that the faster particles are decelerated and the slower particles are accelerated by

the wave. If the fast particles are fewer than the slow particles in resonance, then the wave is damped; see for more detail /8-10/.

We will study the interaction of a one-dimensional packet wave of given amplitude and spectral density ($\Theta(k)$) with a plasma. So long as collisions do not take effect, distortion of the initial distribution function is described by the equation:

$$\frac{\partial f}{\partial t} = \mathcal{L}_1 \frac{\partial}{\partial v_\parallel} \frac{\Theta(v_\parallel)}{|\omega' - \omega/k|_{\omega/k = v_\parallel}} \cdot \frac{\partial f}{\partial v_\parallel}. \qquad (55)$$

In accordance with (55), the resonant particles are redistributed and a plateau is formed on the distribution function (equation (55) similar to the thermal conductivity equation in an inhomogeneous medium). Collisions gradually "round off" the edges of the plateau and a stationary state is established, in which

$$F'(v_\parallel) \cong \frac{6/v_\parallel^2}{\frac{\mathcal{L}_1}{\mathcal{L}} \frac{\Theta(v_\parallel)}{|\omega' - \omega/k|} + \frac{6T}{mv_\parallel^2}} \cdot F(v_\parallel); \quad v_\parallel = \frac{(\underline{v} \cdot \underline{k})}{k}. \qquad (56)$$

Since, as the result of inverse action of the wave the derivative varies mainly and not the distribution function itself, then

$$F'(v_\parallel) \cong -\frac{6/v_\parallel^4}{\frac{\mathcal{L}_1}{\mathcal{L}} \frac{\Theta(v_\parallel)}{|\omega' - \omega/k|} + \frac{6T}{mv_\parallel^2}} \cdot F_m = \frac{F'_m}{1 + \frac{E_0^2}{8\pi NT} \left(\frac{v_\parallel}{\omega_0}\right)^3 \frac{N}{12} \frac{(k)}{2\eta} \frac{\Theta(v_\parallel)}{\lambda}}, \qquad (57)$$

where $\langle k \rangle$ is the mean wave number of the packet, η is the half width and $F_m = (N\sqrt{2\pi}\, T/m)\exp(-mv_\parallel^2/2T)$.

Writing (56) in (47), we obtain:

$$\dot{\varepsilon}_k = \pi \frac{\omega_0^2}{k^2}\omega \frac{F'_m(|E_k|^2/8\pi)}{1 + \frac{E_0^2}{2\pi NT}\left(\frac{v_\parallel}{\omega_0}\right)^3 \frac{N}{12}\frac{\langle k\rangle}{2\eta}\frac{\Theta(v_\parallel)}{\lambda}} \qquad (58)$$

and, summing over all harmonics in the packet, we obtain*:

$$\dot{\varepsilon} = \frac{\omega_0^2}{2\langle k\rangle^2} \frac{F'_m(\omega_0\langle k\rangle)E_0^2/2}{1 + \frac{E_0^2}{8\pi NT}\left(\frac{\omega}{\omega_0\langle k\rangle}\right)^3 \frac{N}{12}\frac{\langle k\rangle}{2\eta}\frac{1}{\lambda}}$$

With $E_0^2/8\pi NT \to 1/N_D$, where $N_D = N(v_\parallel/\omega_0)^3(3\gamma/4)$ is the number of particles in a sphere of radius $v_\parallel/\omega_0$, $\dot{\varepsilon} \to 2\gamma E_0^2/8\pi$, where γ is obtained from the linear theory. With $E_0^2/8\pi NT \to 1$, $\dot{\varepsilon} \simeq \gamma E_0^2/8\pi$. Hence the effective "damping time" of the packet envelope

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* This expression permits of simple interpretation. We represent it in the form $\dot{\varepsilon} = (\dot{\varepsilon})_0/(1 + \tau_1/\tau_2)$. Here $(\dot{\varepsilon})_0$ is the energy absorption to a linear approximation ("Landau damping"), τ_1 is the characteristic time for establishing a local Maxwellian distribution, τ_2 is the characteristic time of the distribution function distortion, under the influence of the field of the wave packet envelope. If $\tau_1 \ll \tau_2$, then collisions succeed throughout in rendering the distribution function Maxwellian, and we obtain the usual Landau damping. With increasing wave amplitude, the distortion brought about becomes so large that the collisions do not achieve the Maxwellian distribution and the damping decrement diminishes. This interpretation permits evaluation of the absorption also in the case of a "monochromatic" wave (for which the quasi-linear approximation is invalid). Let the potential amplitude in the wave be φ. Then the particles responsible for absorption will have velocities in the range

$$\frac{\omega}{k} - \sqrt{\frac{2e\varphi}{m}} < v < \frac{\omega}{k} + \sqrt{\frac{2e\varphi}{m}}$$ (particles reflected from the "potential

* (continuation of footnote to p.42)

humps").

This means that the distribution function is distorted most in a zone with a width in the order of $\sqrt{e\varphi/m}$. On account of Coulomb collisions with small angles of scatter, local equilibrium in this zone will evidently be restored after the time $\tau_1 \sim e\varphi/vT$. The nonlinear distortion time under the action of the wave field is in the order of $\tau_2 \sim \lambda/\sqrt{e\varphi/m}$, where λ is the wavelength. We finally obtain

$$\dot{\varepsilon} \sim (\dot{\varepsilon})_0/(1 + (e\varphi)^{\frac{1}{2}}/T\lambda\sqrt{m}).$$

varies from a time in the order of $1/\gamma$ for a wave not exceeding the thermal noise level, to a time in the order of the collision time for a high-amplitude wave ($E_0^2 \sim 8\pi NT$).

(b) In the presence of a magnetic field

We will limit ourselves to a study of the absorption of a ~~one-dimensional packet~~ of "extraordinary" waves with circular polarisation in the vicinity of frequencies in the order of $\omega_H = eH/mc$.

Since $\omega < \omega_H$, resonance occurs in the zone of negative velocities $v_\parallel$ ($v_\parallel k < 0$). The wave electric field acts on the resonant particles on the one hand, "untwisting" them, and on the other hand the magnetic field, setting up a "radiation" pressure.

Distortion of the particle distribution function f_0 after a time less than the collision time is described by the equation

$$\frac{\partial f_0}{\partial t} = \mathcal{L}_1 \left(\frac{1}{v} \frac{\partial}{\partial v} v + \frac{v_\parallel k}{\omega_H} \frac{\partial}{\partial v} \right) \frac{\Theta(v)}{\omega - (\omega - \omega_H)/k} \cdot \left(\frac{\partial}{\partial v} + \frac{v_\parallel k}{\omega_H} \frac{\partial}{\partial v} \right) f_0 \qquad (60)$$

If instead of $v_\parallel$, $v_\perp$ we introduce the fresh variables

$$w_\parallel = v_\parallel, \quad w = v - \frac{v_\parallel^2}{2v}, \qquad (61)$$

then, in the zone where $\Theta/|\omega' - (\omega - \omega_H)/k|$ can be considered constant, equation (60) is converted to:

$$\frac{\partial f_0}{\partial t} = L_1 \frac{1}{w} \frac{\partial}{\partial w} w \frac{\partial f_0}{\partial w}, \qquad (62)$$

where

$$L_1 = \frac{\Theta}{|\omega' - (\omega - \omega_H)/k|} \mathcal{L}_1$$

Equation (62) describes the equalising process of the distribution function along the lines: $W_{\parallel} = \text{const.}$, i.e. $v_{\parallel} - (v_{\perp}^2/2v_{\varphi}) = \text{const.}$ The effective diffusion coefficient ~~turns to~~ becomes zero at the boundary of the resonance zone with $v_{\parallel} = v_1, v_2$. However, after a time considerably greater than the collision time, a completely different state is established asymptotically. This is evidently governed not by the initial distribution function but by the boundary conditions. It is necessary that on the right of v_1 the distribution function should be Maxwellian and ~~with~~ at $v_{\parallel} = v_1$ there should be no distribution function discontinuity. Therefore, disregarding the effect of collisions, the function is evidently equal to:

$$f_0^0 \quad \frac{N}{(2\pi)^{3/2}(T/m)^3} \exp\left\{ -\frac{mv_{\perp}^2}{2T} - \frac{mv_{\varphi}v_{\parallel}}{T} + \frac{mv_{\varphi}v_1}{T} - \frac{mv_1^2}{2T} \right\} \qquad (63)$$

The function f_0^0 represents a zero approximation according to $\mathcal{L}/\mathcal{L}_1$, that is to say according to the inverse wave amplitude. In obtaining this expression the "tail" of the Maxwell distribution is disregarded ($v_{\parallel} < v_2$). Since, according to the condition, $v_{\varphi} \gg |v_{\parallel}|$, then, assuming $mv_{\varphi}\Delta v/T \gg 1$, where $\Delta v = v_1 - v_2$, the "tail" particles can be disregarded. Expanding the real distribution function in series in $\mathcal{L}/\mathcal{L}_1$ we obtain from (52) the expression:

$$\cdot \left(\frac{\partial f_0^1}{\partial v} + \frac{v_{\perp}}{v_{\varphi}} \frac{\partial f_0^1}{\partial v_{\parallel}} \right) - \frac{2m^2 v_{\varphi}^2}{T^2} \frac{\mathcal{L}}{\mathcal{L}_1} \left(\frac{v_{\perp}^2 + 2v_{\parallel}^2}{v} - 2v_{\parallel} \right) f_0^0 \qquad (64)$$

Thence, generally speaking, the first correction to the distribution function f_0^1 can readily be found; this, however, is not of interest here.

Writing expression (64) in (54), we obtain the energy flux from the k-th component of the wave to the plasma:

$$\frac{8\pi\gamma_k}{H_0^2} = -\frac{8\pi NT}{H_0^2}\lambda\frac{1}{N_D}\left(\frac{\omega_k}{\omega_k-\omega_H}\right)^2\omega_0\frac{2\eta}{(k)} \times\exp\left\{-\frac{mv_\parallel^2}{2T}+\frac{m}{T}v_\varphi(v_\parallel-v_1)\right\} \quad (65)$$

where 2η is the width of the packet, and $H_k^2 = H_0^2\Theta/2\eta$

N_D is the number of particles in a "Debye" sphere,

$v_\parallel$ is the resonant particle velocity and v_1 is the resonance boundary.

If the packet is sufficiently large, then the effective width of the zone (in the k space), within which energy-exchange occurs is $2\eta^* \sim (T/mv_\parallel^2)\left[(\omega_H - \omega/v_\varphi)\right]$ (where v_φ is the phase velocity of the wave).

Near the boundary of the resonance zone

$$\gamma_k = -\frac{8\pi NT}{H_0^2}\lambda\cdot 3\sqrt{2\pi}\,\frac{1}{N_0} \times\left(\frac{\omega_k}{\omega_k-\omega_H}\right)^2\omega_0\frac{2\eta}{(k)}\,e^{-mv_1^2/2T} \quad (66)$$

At the same time, from equation (54), substituting the undistorted Maxwellian electron distribution, we obtain the known result of the linear theory /18/:

$$\gamma^0 = \frac{\omega_H - \omega}{\omega_H}\sqrt{\frac{\pi}{2}}\frac{\omega_0^2}{k\sqrt{T/m}}\exp\left[-\frac{m(\omega_H-\omega)^2}{2Tk^2}\right]$$

$$\gamma_k = \gamma^0\beta - \frac{8\pi NT}{H_0^2} = \frac{N_D}{8}\frac{\omega_0}{2\eta\sqrt{T/m}}\left(\frac{\omega_H-\omega_0}{\omega_H}\right)^2\beta_0$$

In actual fact, this equation, like our entire approximation, is certainly invalid with $\beta \sim \beta_0$. The function $\gamma(1/\beta)$ according to (66) is shown by the broken line in Fig. 16. The true γ_k curve is shown by the solid line.

On account of the sharp cut-off of the distribution function f_0 the total energy transfer from the packet to the plasma is still more sharply dependent on β.

6. Stability of nonlinear motions in rarefied plasma

It is known that a plasma which is sufficiently removed from a state of thermodynamic equilibrium (in other words plasma with a non-Maxwellian velocity distribution), is unstable with respect to self-excitation of different types of oscillation.

By way of example we can mention so-called instability in the presence of a beam of particles when the particle velocity distribution function possesses secondary maxima, or instability occurring when the distribution function is sufficiently anisotropic.

It is clear that the propagation of sufficiently large perturbations in rarefied plasma must be accompanied by marked deviation of the plasma from thermodynamic equilibrium. As an extreme case, "multiflux" motion may serve here /20-21/, occurring as the result of "inversion" of waves with amplitudes exceeding some critical value.

6.1. Instabilities of multicomponent motion

If the characteristic dimensions of multicomponent motion zones are substantially in excess of the length of the waves of the resulting instabilities, use can be made of the results of the instability investigation in the case of homogeneous plasma.

For example, in the case of two opposite ion fluxes travelling across a magnetic field at velocities v_0 and $-v_0$, the dispersion equation takes the form /19/:

$$\frac{\omega_H^2}{\omega_H \Omega_H} \quad \frac{1}{(\omega - k v_0)^2} + \frac{1}{(\omega + k v_0)^2}$$

The maximal instability increment is equal, in order of magnitude, to $(\omega_H \Omega_H)^{\frac{1}{2}}$. For $v_0 \lesssim H/\sqrt{4\pi n_0 M}$ the characteristic instability wavelength is in the order of c/ω_0. Hence, multicomponent motions across a magnetic field (section 3) are unstable.

Again in the case of two ion fluxes travelling counter to each other in the absence of a magnetic field (the electrons are now assumed "hot" and with a Bolzmann spatial distribution $n = n_0 \exp(e\varphi/T)$) the dispersion equation takes the form:

$$\frac{1}{(\omega + k v_0)^2} + \frac{1}{(\omega - k v_0)^2} \quad \frac{2M}{k^2 T}$$

It is curious that, in this case, sufficiently fast ion fluxes ($v_0 > \sqrt{T/M}$) are stable*.

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* At very large values $v_0 (> \sqrt{T/M})$ instability again occurs, but on the "electron" branch.

6.2. Instabilities of waves of finite amplitude propagated across a magnetic field

Instabilities in waves of finite amplitude can, in principle, occur even before the emergence of "multicomponent" motion, that is to say with smaller wave amplitudes. We will consider, for example, a stationary solitary wave travelling across a magnetic field in "cold" plasma ($8\pi n T \ll H^2$). We will return to the picture of plasma ion and electron motion in such a wave (Fig. 17). If $H^2 \ll 8\pi n m c^2$, the plasma is quasi-neutral. In the direction of wave propagation both the ions and electrons travel at the same velocities. However, the electrical current in the direction perpendicular to the wave velocity and magnetic field is made up only of electrons. In homogeneous plasma, of course, the presence of a marked ion and electron relative motion leads to instability. It is clear that an analogous effect can be anticipated here also. The problem is simplified if, when studying small deviations from the stationary picture of the solitary wave, the terms taking account of unperturbed plasma motion in the x direction are disregarded. Obviously this is valid when the instability development time is considerably shorter than the time taken for the plasma to "traverse" the solitary-wave zone. This latter is equal in order of magnitude obviously to $L/(H/\sqrt{4\pi n_0 M})$, where L is the wave "width".

The perturbed motion of ions and electrons will be considered, for simplicity, as the motion of two "liquids" with adiabatic laws of pressure variation*.

* In homogeneous plasma, this approach yields qualitatively correct results.

The influence of the magnetic field on perturbed motion will not be considered, confining the case to oscillation frequencies substantially higher than the Larmor frequency of electrons eH/mc. To this approximation, the equations for the perturbed quantities: v_x, v_y (electron velocity component), V_x, V_y (ion velocity components), n_e, n_i (electron and ion densities) and φ (electrical potential) take the form

$$i(\omega + kv_0)v_y = ik\frac{e}{m}\varphi - ik\frac{s_0^2}{n_0}n, \quad (67)$$

$$i(\omega + kv_0)v_x = -\frac{e}{m}\frac{d\varphi}{dx} - \frac{s_0^2}{n_0}\frac{dn}{dx}, \quad (68)$$

$$i\omega V_y = -ik\frac{e}{M}\varphi, \quad (69)$$

$$i\omega V_x = -\frac{e}{M}\frac{d\varphi}{dx}, \quad (70)$$

$$i(\omega + kv_0)n_e + ikn_0v_y + \frac{d}{dx}(n_0v_x) = 0, \quad (71)$$

$$i\omega n_i + \frac{d}{dx}(n_0V_x) + ikn_0V_y = 0 \quad (72)$$

$$-k^2\varphi + \varphi'' = 4\pi e(n_e - n_i). \quad (73)$$

Here (67)-(70) are the equations of motion for electrons and ions and (71)-(72) is the continuity equation for electrons and ions, whilst finally (73) is the Poisson equation for the electric field. We immediately select the perturbed quantities in the form $\varphi(x)\exp(i\omega t + iky)$. v_0, s_0^2 and n_0 entering into the equations represent the unperturbed electron mean velocity (along the y axis), the square of the electron thermal velocity and the plasma density as functions of x. On the assumption that the derivatives

~~for x from~~ of the perturbed ~~values~~ quantities with respect to x are considerably greater than the derivatives ~~from~~ of the unperturbed quantities ("quasi-classical approximation")

the system of equations (67)-(73) is reduced to a single second-order differential equation for n_e:

$$n_0^2 \frac{d^2 n_e}{dx^2} + \left[(\omega + kv_0)^2 - s_0^2 k^2 - 1 - \frac{\omega_0^2}{(\Omega_0^2/\omega^2)}\right] n_e = 0 \quad (74)$$

The stability investigation is reduced to a problem on the ~~specific~~ eigen values of equation (74). Local solutions are involved, attenuating on both sides of the solitary wave. We will investigate the behaviour of the function

$$F(x, \omega, k) \quad (\omega + kv_0)^2 - s_0^2 k^2 - 1 \quad \frac{\omega_0^2}{(\Omega_0^2/\omega^2)} \quad (75)$$

In homogeneous plasma, this function would not be dependent on x and the dispersion equation connecting ω with k would take the form

$$F(\omega, k) \quad 0 \quad (76)$$

This equation yields unstable solutions when $v_0^2 > s_0^2$, that is to say when the mean velocity of relative motion of the ions and electrons exceeds the thermal velocity of the electrons. For not too high values of k ($k^2 \ll \omega_0^2/s_0^2$), this equation can be written approximately as

$$F(\omega, k) \approx k^2 (v_0^2 - s_0^2) - 1 \quad \frac{\omega_0^2}{(\Omega_0^2/\omega^2)} \quad 0 \quad (77)$$

Then we find

$$\omega^2 = \frac{\Omega_0^2 k^2 (s_0^2 - v_0^2)}{\omega_0^2 - k^2 (v_0^2 - s_0^2)} \quad (78)$$

with $v_0^2 > s_0^2$, ω becomes imaginary (instability). Returning to the inhomogeneous problem, we will study the spatial variation of the function $F(x, \omega, k) = k^2(v_0^2 - s_0^2) - \omega_0^2/(1 - \Omega^2/\omega^2)$. (To this approximation it is sufficient to consider the real ω^2.)* For demonstration we show (Fig. 18) the variation profile of v_0 and ω_0^2* in the solitary wave as a function of x. In the zone $v_0^2 > s_0^2$, where $F(x, \omega, k) > 0$, for n_e there is an oscillating solution. Far from the solitary wave $F(x, \omega, k) < 0$, corresponding to exponentially attenuating solutions. At the "turning" points, where $F(x, \omega, k) = 0$, these solutions merge. Thus, the necessary local solutions always exist, whilst instability occurs if there is a zone where $v_0^2 > s_0^2$ within the solitary wave. The growth increments of such instability, as is known, are in the order of Ω_0 (in "zero" temperature plasma the maximal increment is still greater, $\Omega_0(M/m)^{1/6}$).

The amplitude v_0^2 in the solitary wave increases with increasing Mach number ($\underline{M} = u\sqrt{4\pi n_0 M/H_0}$). Finally, for a certain Mach number $\underline{M} = \underline{M}^*$ the amplitude v_0^2 exceeds the mean thermal velocity of electrons, that is to say the wave becomes unstable. From the results of Section 4 $\underline{M}^*$ can be readily evaluated for "cold" plasma ($8\pi n T \ll H^2$):

$$\underline{M}^* \approx 1 + \frac{3}{8}\left(\frac{8\pi n T}{H^2}\right)^{\frac{1}{3}}; \qquad (79)$$

---

* The dependence of s_0^2 on x provides the relationship of ω_0^2 on x.

The influence of this type of instability can, in a certain sense, be treated as the frictional force of electrons on ions, possessing a "collective" nature. It should be finally observed that, although the instability considered was applied only to solitary stationary waves, the same situation can occur for any given waves in plasma with a magnetic field.

In the conditions ($nT \sim H^2/8\pi$) instabilities of another type would apparently be more dangerous, the occurrence of which can be represented as follows. At points of "constriction" of the magnetic field in the wave, the energy of thermal motion of electrons increases adiabatically in a plane perpendicular to the magnetic field ($T_\perp$), whilst the "longitudinal" energy ($T_\parallel$) varies only slightly or remains constant. With $T_\perp/T_\parallel$ exceeding some critical value, it is natural to expect build-up of oscillations with a frequency in the order of the cyclotron frequency of electrons (as occurs in homogeneous plasma)/18/.

Captions

Fig. 1 Acoustic wave-form distortion due to non-linearity.

Fig. 2 Frequency as a function of wave vector for magneto-acoustic oscillations.

Fig. 3 Particle displacement ~~Frequency shift~~ in nonlinear Langmuir oscillations.

Fig. 4 Two-velocity ~~component~~ current ahead of the piston.

Fig. 5 Formation of the three-velocity current zone.

Fig. 6 Two-velocity current across a magnetic field.

Fig. 7 Integral curves on the phase plane.

Fig. 8 Solitary ion-acoustic wave.

Fig. 9 System of co-ordinates for a wave travelling across the field.

Fig. 10 Solitary wave: (a) - compression; (b) - rarefaction.

Fig. 11 Distortion of the distribution function.

Fig. 12 Absorption of Langmuir oscillations as a function of amplitude.

Fig. 13 Particle reflection from the potential "hump".

Fig. 14 Resonant particles for the extraordinary wave.

Fig. 15 Characteristics on the plane ($v_\perp$, $v_\parallel$).

Fig. 16 Absorption of the extraordinary wave as a function of amplitude.

Fig. 17 Solitary wave.

Fig. 18 Velocity and density variation in a solitary wave.

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Manuscript received 13th January 1961

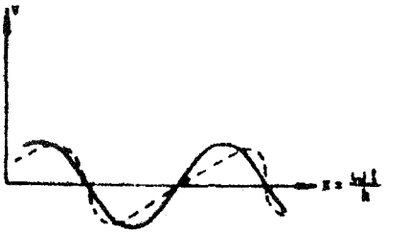

Рис. 1 Искажение формы звуковой волны из-за нелинейности.

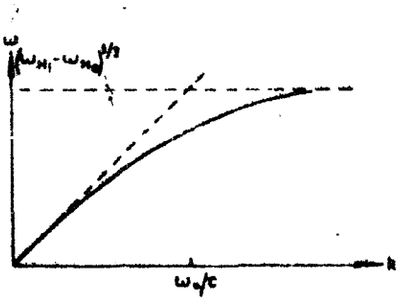

Рис. 2 Зависимость частоты от волнового вектора для магнитозвуковых колебаний.

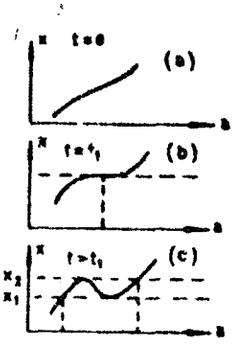

Рис. 3 Смещение частицы в нелинейных ленгмюровских колебаниях.

Рис. 4 Двухскоростное течение перед поршнем.

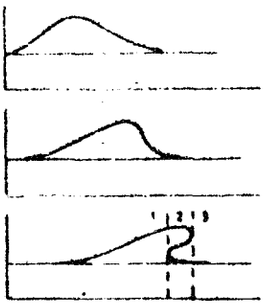

Рис. 5 Образование области трехскоростного течения.

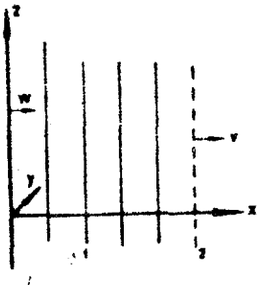

Рис. 6 Двухскоростное течение поперек магнитного поля.

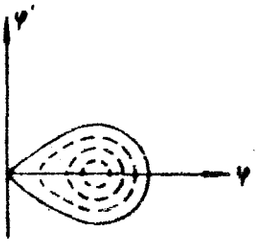

Рис. 7 Интегральные кривые на фазовой плоскости.

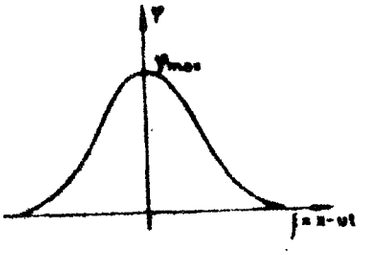

Рис. 8 Уединенная ионно-звуковая волна.

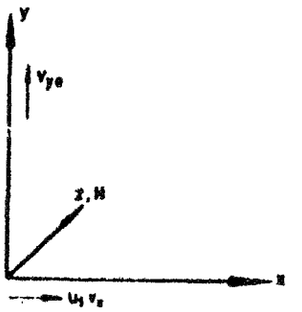

Рис. 9 Система координат для волны, распространяющейся поперек поля.

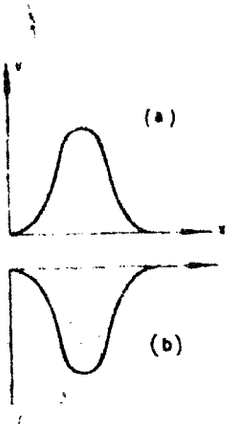

Рис. 10 Уединенная волна (a) сжатия; (b) разрежения

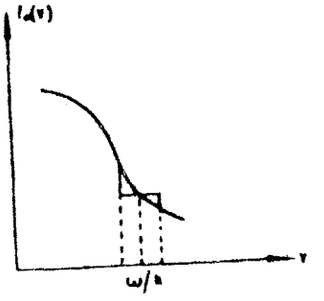

Рис. 11 Искажение функции распределения.

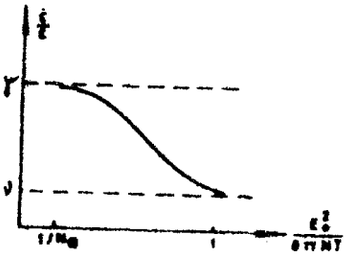

Рис. 12 Зависимость поглощения ленгмюровских колебаний от амплитуды.

Рис 13 Отражение частиц от потенциального «горба»

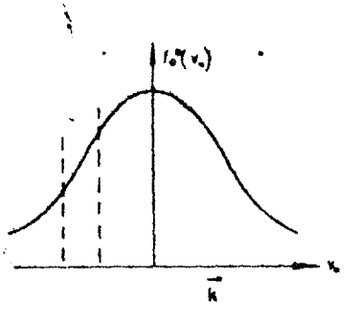

Рис 14 Резонансные частицы для необыкновенной волны

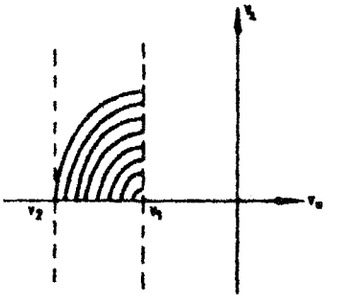

Рис. 15 Характеристики на плоскости (,).

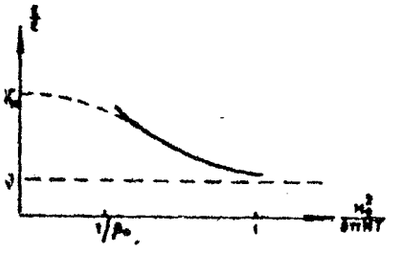

Рис. 16 Зависимость поглощения необыкновенной волны от амплитуды.

Рис. 17 Уединенная волна.

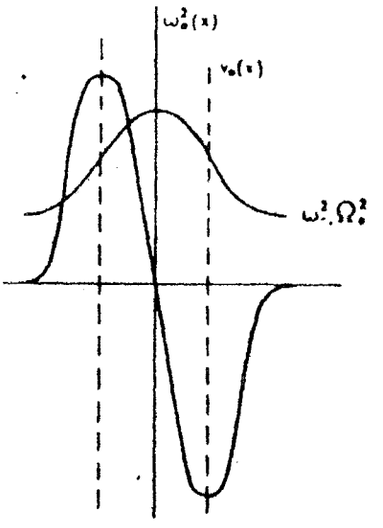

Рис 18 Изменение скорости и плотности в уединенной волне.